Complexity of time series associated to dynamical systems inferred from independent component analysis

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A not trivial problem for every experimental time series associated to a natural system is to individuate the significant variables to describe the dynamics, i.e., the effective degrees of freedom. The application of independent component analysis (ICA) has provided interesting results in this direction, e.g., in the seismological and atmospheric field. Since all natural phenomena can be represented by dynamical systems, our aim is to check the performance of ICA in this general context to avoid ambiguities when investigating an unknown experimental system. We show many examples, representing linear, nonlinear, and stochastic processes, in which ICA seems to be an efficacious preanalysis able to give information about the complexity of the dynamics.

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I. INTRODUCTION

Very general, natural systems can be considered dynamical systems, whose evolution can be studied looking at time series associated to significant variables on a suitable scale. These series are often characterized by a strong spatial coherence. Independent component analysis (ICA) seems to be the suitable technique able to study their level of complexity [1]. Namely, we have applied ICA in seismological and atmospheric context, getting information about the complexity of the dynamical systems involved in the effective dynamics [2–5]. Our goal is now to check the performance of ICA in an abstract context to avoid ambiguities, as much as possible, when investigating an unknown experimental system. This paper is the first effort along this line.

We consider dynamical systems (DSs) associated to sequences of observations (measurements) made in the course of the time, but we remark that the scheme of DSs is very general, namely sequences can be generated by every type of phenomena. For example, we can consider the series produced by a computer via a given calculus device or the alphabetical words in a text and so on. We can associate, in a natural way, the concepts of complexity, statistics, and ergodicity to sequences in order to quantitatively distinguish them. In this context, the theoretical scheme of dynamical systems becomes the unifying theory. There are many tools giving powerful methods to study asymptotic properties, but these methods require analytical solutions, i.e., infinite sequences. In the real experimental cases, we have to extract all the properties considering finite series. Hence numerical analysis arises to understand all important parts of information available and included in the analyzed sequences.

In the cases when the same signal contains information relative to different DSs or to the many degrees of freedom of the underlying dynamical system, if the systems are linearly coupled or the degrees of freedom are uncoupled, we should recognize them as independent components in the experimental signals. There are many numerical methods used for this aim (see, e.g., [6]). But ICA could constitute a pregnant preanalysis.

We select different DSs representative of large classes: linear, piecewise linear, and nonlinear in the regime of limit cycle, chaotic, and stochastic and we study the ICA performance. In Sec. II, we summarize the main features of ICA, while in Sec. III we describe the DSs and the corresponding experiments for linear, piecewise linear, nonlinear, and stochastic systems. Finally, conclusions follow.

II. INDEPENDENT COMPONENT ANALYSIS

ICA is a method to find underlying factors or components from multivariate (multidimensional) statistical data, based on their statistical independence. ICA was introduced in the early 1980s by Hérault and Ans [7] and developed for problems closely related to the cocktail party problem. Several different implementations of ICA can be found in literature but the algorithm, which has contributed to the application of ICA to large scale problems for its easy implementation and mainly for its computational efficiency, was introduced by Hyvärinen and Oja, i.e., the *fixed-point* FastICA algorithm [8]. Since then, ICA has revealed many interesting applications in different fields of research (biomedical signals, geophysics, audio signals, image processing, financial data, etc.).

In its simplest form, ICA performs a blind separation of statistically independent sources, assuming linear mixing of

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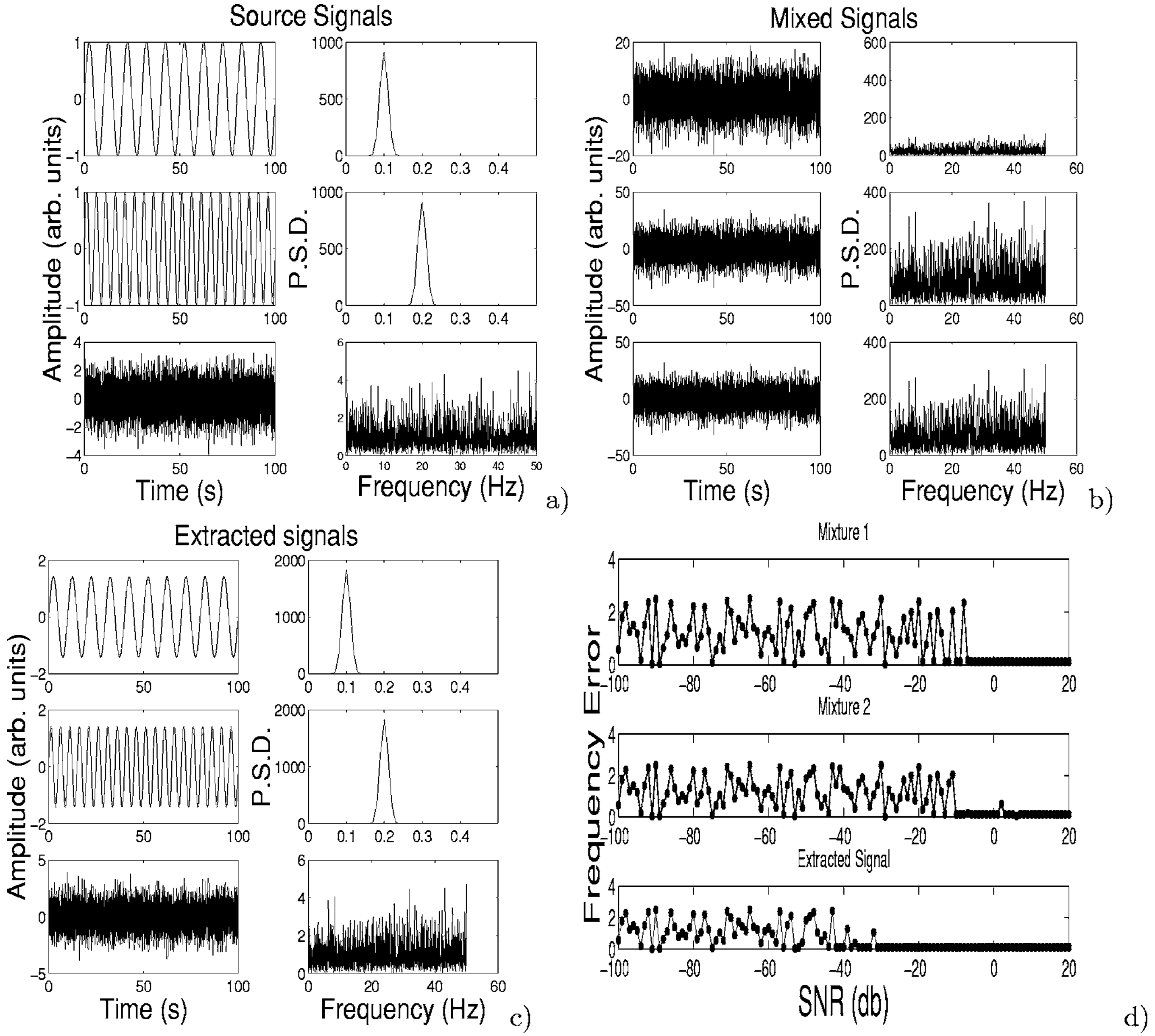

FIG. 1. Separation of a mixture of two harmonic oscillators and Gaussian noise: (a) the source signals (corresponding to the two linear oscillators and noise); (b) mixed signals; (c) extracted signals via ICA; and (d) performance of ICA in extracting periodic signals from noise, varying the signal-to-noise ratio (SNR) from −100 to 20 db. The frequency error measures the difference between the true frequency and the estimated one computing PSD. It is remarkable that the mere application of PSD on the mixtures fails at SNR equal to −10 db, while ICA allows the extraction of the periodic signals from noise with SNR equal to −40 db.

the sources at the sensors. The intuitive notion of maximum non-Gaussianity is used in ICA estimation adopting techniques which involve higher-order statistics. We remind that classical measures of non-Gaussianity are the kurtosis, the negentropy, and the mutual information.

We assume an instantaneous mixing model, thus we neglect any time delay that may occur in the mixing. Formally, the mixing model is written as

$$\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{n} \tag{1}$$

where $\mathbf{x}$ is an observed m-dimensional vector, $\mathbf{s}$ is an n-dimensional random vector whose components are assumed to be mutually independent; $\mathbf{A}$ is a constant $m \times n$ matrix to be estimated, and $\mathbf{n}$ is the additive noise. The additive noise term $\mathbf{n}$ is often omitted in Eq. (1) because it can be incorporated in the sum as one of the source signals. In addition to the independent assumption, we assume that the number of available different mixtures m is at least as large as the number of sources n. Usually, m is assumed to be known in advance, and often $m=n$ (there exists a probabilistic version of ICA that allows one to bypass this limit [9]). Only one of the source signals s_i is allowed to have a Gaussian distribution because it is impossible to separate two or more Gaussian sources [10,11]. On the extract components by using ICA there are the following ambiguities.

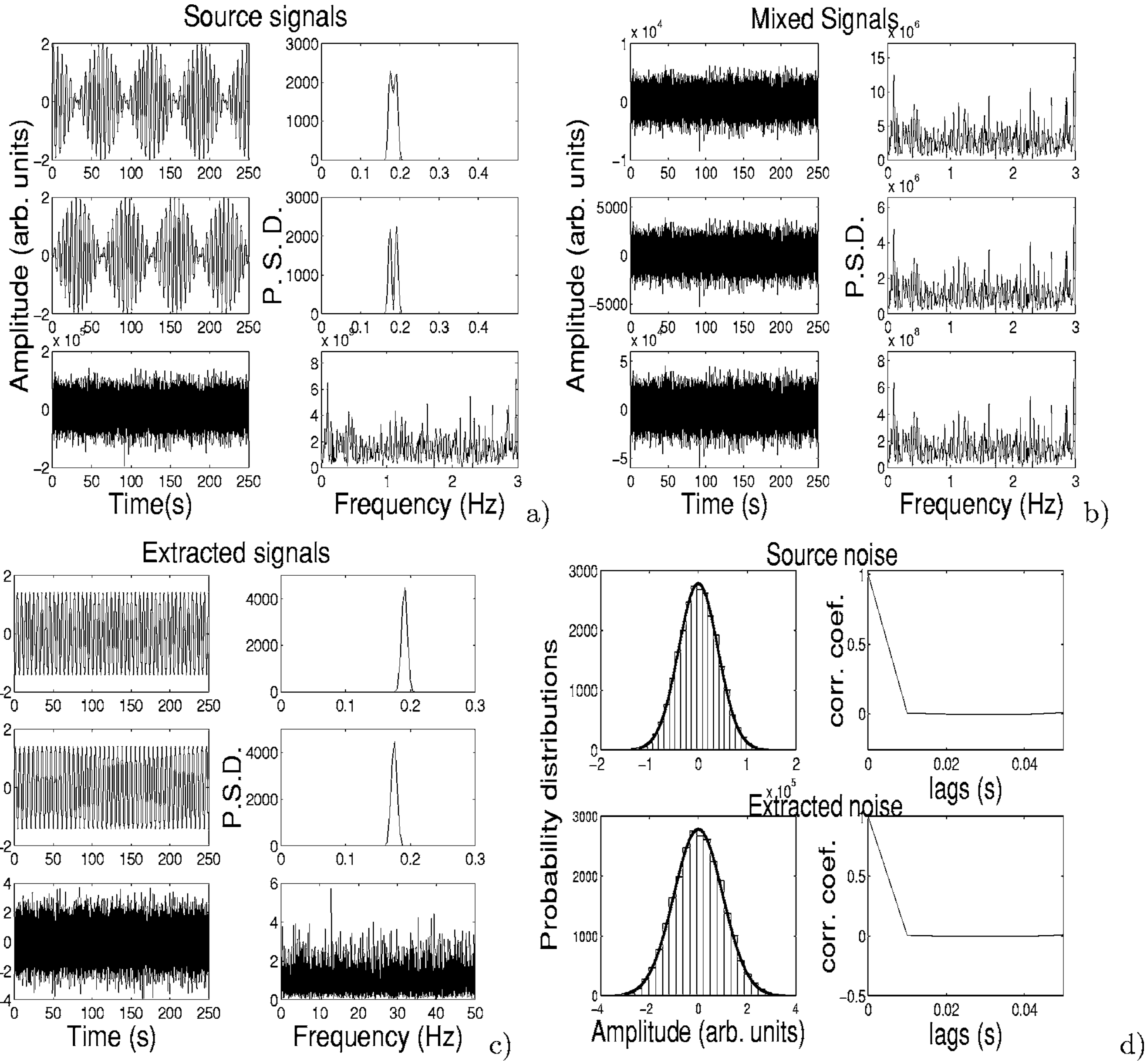

FIG. 2. Separation of the mixture of two coupled harmonic oscillators in beat regime and Gaussian noise (related spectra are on the right): (a) source signals; (b) mixed signals; (c) extracted signals; (d) distribution function and autocorrelation of source and extracted noise.

(1) We cannot determine the variances of the independent components. Very little, *a priori*, information is assumed on matrix **A**, therefore, the magnitudes of the basis vectors of the matrix **A** and the amplitudes of the source signals can be interchanged in Eq. (1). To get a unique expansion, the most natural way is to assume that each source has unit variance. Then the matrix **A** will be adapted in the ICA solution methods to take into account this restriction. This still leaves the ambiguity of the sign.

(2) We cannot determine the order of the independent components. In adaptive source separation an $m \times n$ separating matrix **B** is updated so that the vector

$$\mathbf{y} = \mathbf{B}\mathbf{x} \tag{2}$$

is an estimate $\mathbf{y} \simeq \mathbf{s}$ of the original independent source signals.

Some heuristic approaches have been proposed in literature to achieve the separation. Among them, a good measure of independence is given by negentropy J. It is based on the information-theoretic quantity of differential entropy **H** of a random vector **y** with density $f(\cdot)$ and it is defined as follows:

$$J(\mathbf{z}) = H(\mathbf{z}_{gauss}) - H(\mathbf{z}) \tag{3}$$

where $\mathbf{z}$ is a random variable and $\mathbf{z}_{gauss}$ is a Gaussian random variable of the same covariance matrix as $\mathbf{z}$. The estimate of negentropy is difficult and, in practice, some approximations must to be introduced. In the following we shall use the *fixed-point* algorithm, namely FastICA [1]. Rigorously, this algorithm is based on an approximative Newton iteration scheme. The FastICA learning rule finds a direction, i.e., a unit vector $\mathbf{w}$ such that the projection $\mathbf{w}^T\mathbf{x}$ maximizes inde-

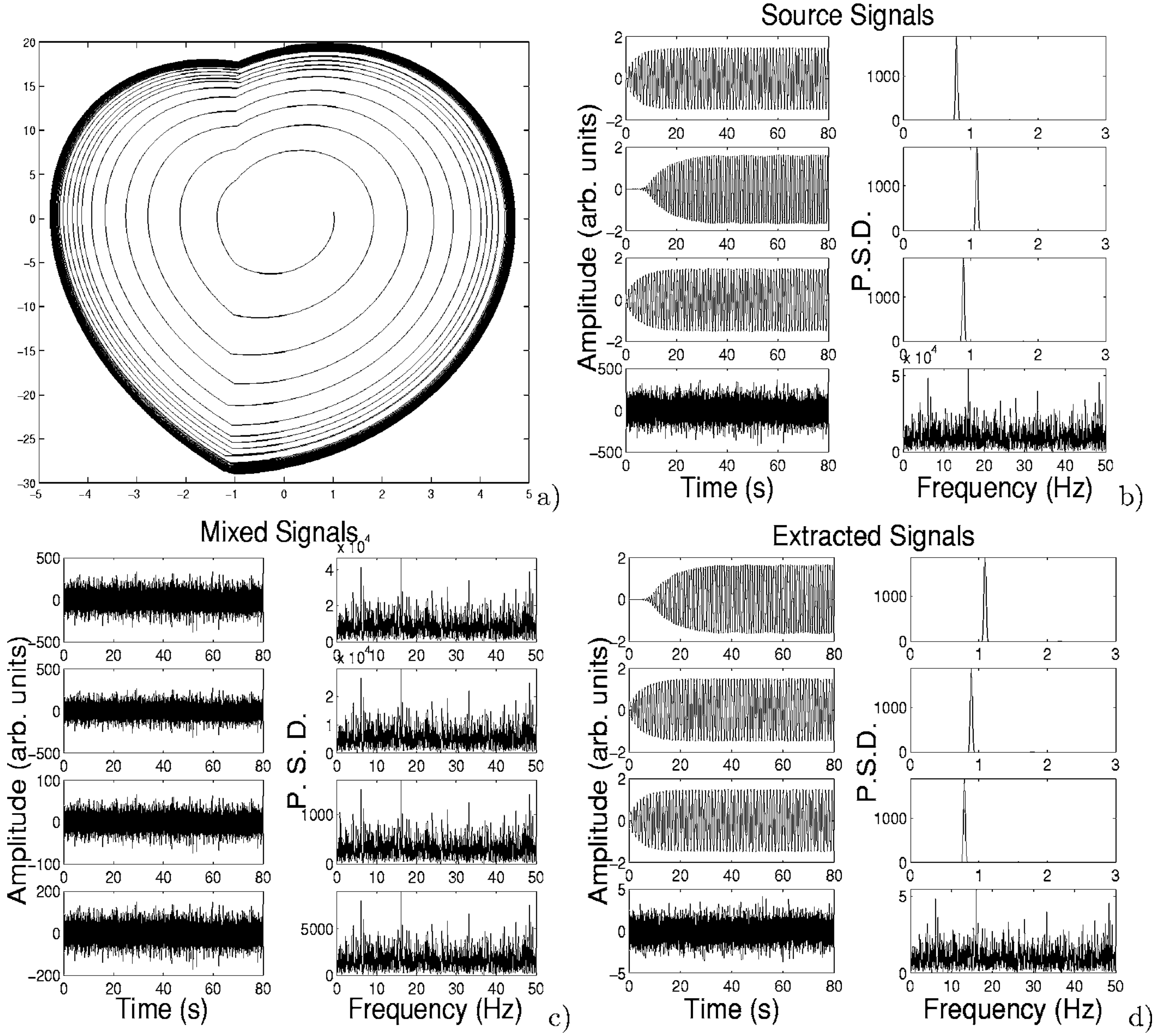

FIG. 3. Separation of mixtures of three self-sustained oscillators and Gaussian noise: (a) representation of the Andronov oscillator in the phase space; (b) source signals and their spectra; (c) mixed signals and their spectra; (d) extracted signals and relative spectra.

pendence of the single estimated source y. Independence is here measured by the approximation of the negentropy given by

$$J_G(\mathbf{w}) = [E\{G(\mathbf{w}^T\mathbf{x})\} - E\{G(\nu)\}]^2, \quad (4)$$

where G is a suitable contrast nonquadratic function, $\mathbf{w}$ is a m-dimensional (weight) vector, $\mathbf{x}$ represent our mixture of signals, and $E\{(\mathbf{w}^T\mathbf{x})^2\}=1$, ν is a standardized Gaussian random variable. Maximizing J_G allows one to find *one* independent component. We remark that the algorithm requires a preliminary whitening of the data (let us define $\mathbf{v}$). Whitening can always be accomplished by, e.g., principal component analysis [1].

The one-unit *fixed-point* algorithm for finding a row vector $\mathbf{w}$ is [1]

$$\mathbf{w}^* = E[\mathbf{v}g(\mathbf{w}_i^T\mathbf{v})] - E[g'(\mathbf{w}_i^T\mathbf{v})]\mathbf{w}_i,$$

$$\mathbf{w}_i = \mathbf{w}_i^*/\|\mathbf{w}_i^*\|, \quad (5)$$

where $g(\cdot)$ is a suitable nonlinearity, in our case $g(y) = \tanh(y)$, and $g'(y)$ is its derivative with respect to y.

The algorithm of the previous equations estimates just one of the independent components. To estimate several independent components, we need to run the one-unit FastICA algorithm using several units (e.g., neurons) with weight vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$. To prevent different vectors from converging to the same maximum we must decorrelate the outputs $\mathbf{w}_1^T\mathbf{x}, \ldots, \mathbf{w}_n^T\mathbf{x}$ after each iteration. In specific applications it may be desired to use a symmetric decorrelation in which

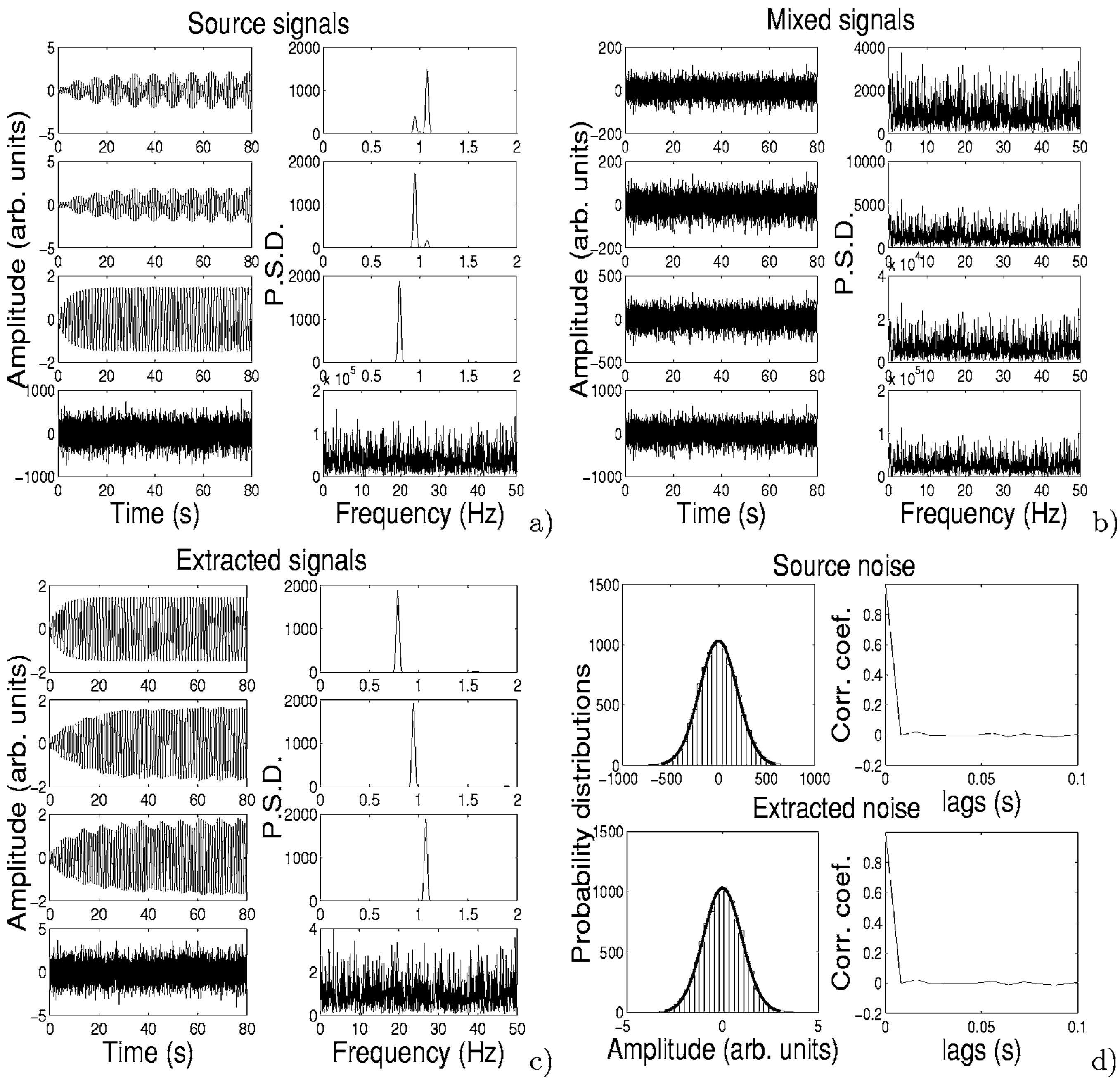

FIG. 4. Separation of mixtures of two coupled Andronov oscillators, one Andronov oscillator, and Gaussian noise: (a) source signals; (b) mixed signals; (c) extracted signals; and (d) distribution function and autocorrelation of source and extracted noise.

vectors are not privileged over the others. This can be accomplished by the classical method involving matrix square roots.

III. DYNAMICAL SYSTEMS

We consider many types of DSs: linear, piecewise linear, and nonlinear in the regime of limit cycle, chaotic, and stochastic systems, taking into account both DSs with few and infinite degrees of freedom. We have selected these systems both because they are frequently used to describe natural phenomena and because they are inserted in a well consolidated and consistent theoretical framework.

In general, linear or nonlinear systems are described by a set of first order differential equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t)) \tag{6}$$

where $\mathbf{F}(\mathbf{x}(t))$ is a linear or nonlinear field.

The DSs with infinite degrees of freedom that we consider are described by the following Ito equation:

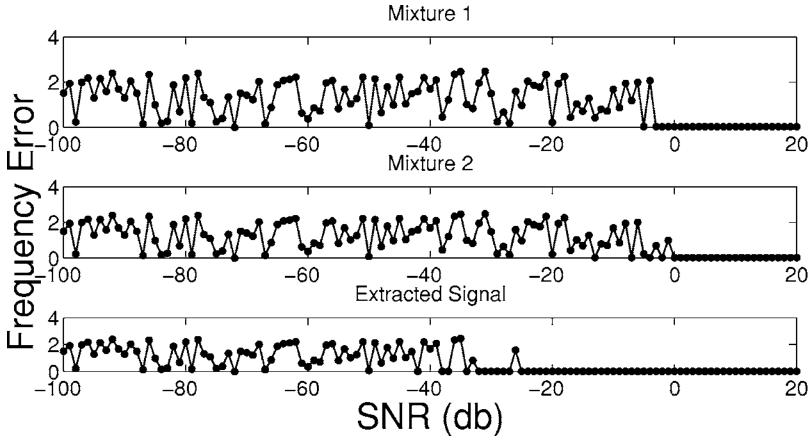

FIG. 5. Frequency estimation: evaluation of the performance of the ICA to separate an Andronov oscillator from Gaussian noise.

$$dx = [v(x,t)]dt + \varepsilon dW, \quad (7)$$

where $v(x,t)$ is a field (drift) and dW is the Wiener process. Now we give a detailed description of the different systems and of the associated experiments.

A. Linear dynamical systems

The linear systems that we consider are single and coupled harmonic oscillators. It is enough to consider two coupled oscillators because the behavior of a system with many oscillators is completely equivalent to that with two coupled ones. We remark that the harmonic oscillator is the paradigm of the linearity and describes all physical systems in the weak coupling limit, when perturbation theory is applicable.

The system of differential equations that describes the motion law of coupled oscillators is well-known. The solution is every linear superposition of the normal modes, depending on initial conditions. Let us describe the results obtained applying ICA to these systems.

1. Experimental results: Linear systems

We have made different experiments to show:

(1) the separation of two harmonic oscillators and additive Gaussian noise;

(2) the separation of two coupled oscillators in beat regime and a Gaussian noise; and

(3) the performance of ICA to separate these kinds of signals.

We remark that in all the cases we use a random matrix with a uniform distribution in the range (0,1) to obtain the mixtures of the signals. In the first experiment, the frequencies (f) of the oscillators are 0.1 and 0.2 Hz, respectively, and the sampling frequency (f_s) is 100 Hz.

In Figs. 1(a)–1(c), the source signals, the mixtures, and the ICA extracted signals are reported together with the rela-

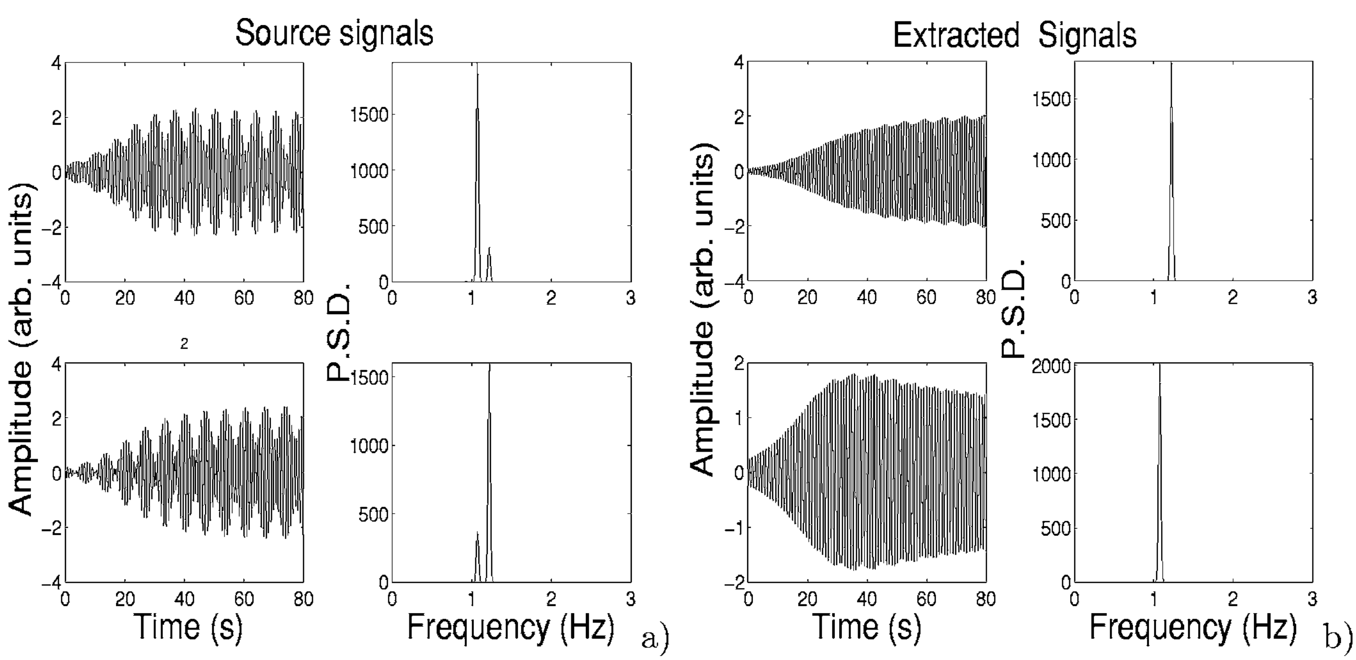

FIG. 6. Separation of limit cycles wave forms from a linear combination of two Van der Pol oscillators: (a) source signals and their spectra; and (b) extracted signals and their spectra.

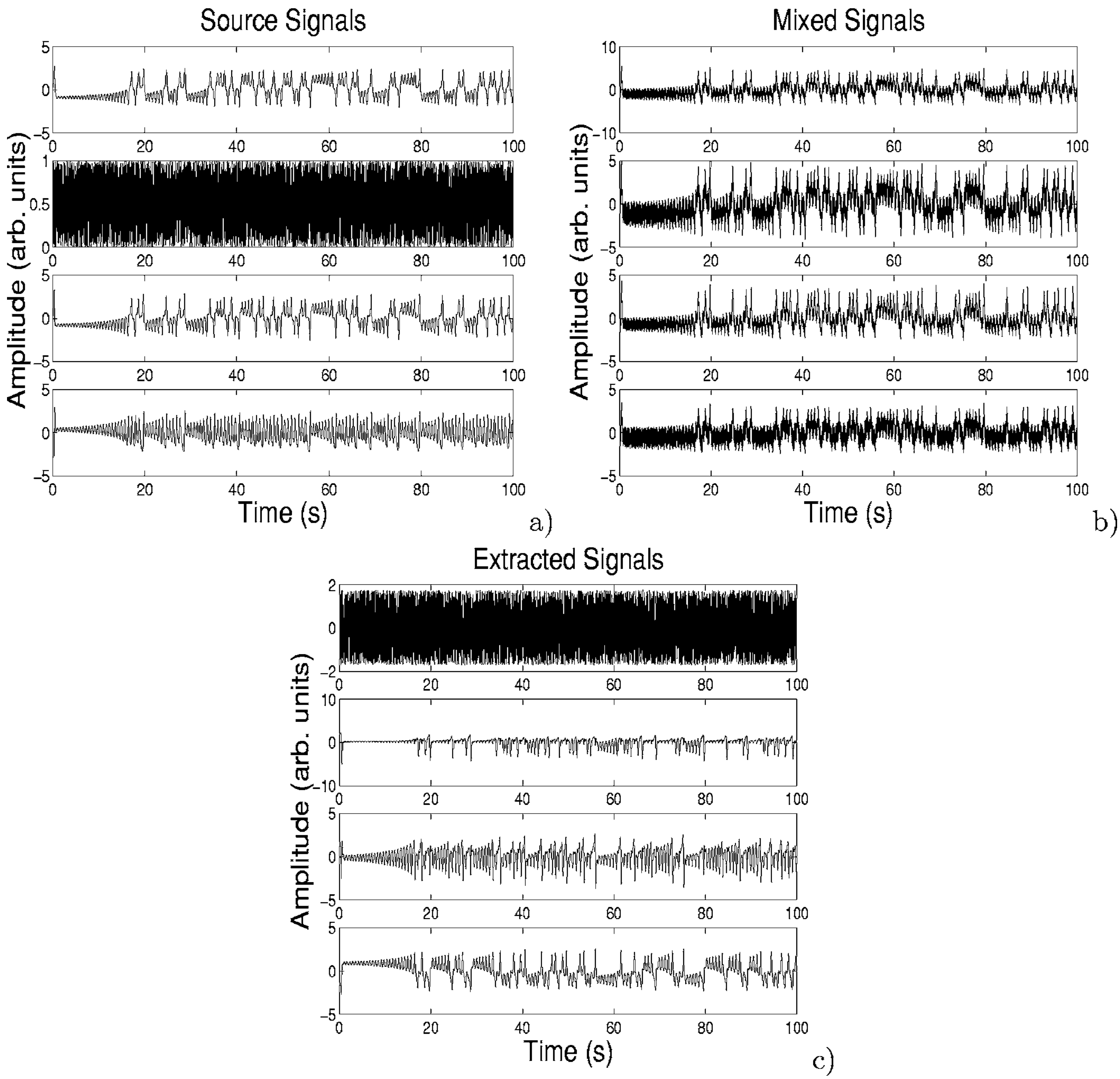

FIG. 7. Separation of signals from the mixtures of the three components of Lorenz oscillator and noise with uniform distribution: (a) source signals; (b) mixed signals; (c) extracted signals.

tive spectra. As it can be seen, the separation is optimal, i.e., the correlation coefficient between the original and the extracted signals is very high (about 0.98).

In Fig. 1(d), the performance of the ICA to separate a harmonic oscillator from a Gaussian noise is shown. In this case, we use a harmonic oscillator with frequency 0.12Hz and we estimate the differences between the true frequency and the estimated one (frequency error) by varying the signal-to-noise ratio (SNR) from −100 to 20 db.

We compute the power spectrum density (PSD) for the two mixtures and for the ICA extracted signals. The estimated frequency corresponds to the maximum peak in the PSD. It is possible to note that the ICA allows one to extract the periodic signal with the proper frequency in time domain from a noise with SNR lower than one achieved by applying PSD on the mixtures [Fig. 1(d)].

In the second experiment, we consider coupled harmonic oscillators in beat regime added to a Gaussian noise. In Fig. 2(a) the time evolution and the relative spectra are plotted. Three different mixtures are shown in Fig. 2(b). Applying ICA, we obtain three extracted signals [Fig. 2(c)], in which the two independent periodic signals are extracted from the noise. We note that, analyzing the PSD of the mixed signals, we do not have the estimated frequencies [Fig. 2(b)], instead analyzing the PSD of the separated signals we obtain the normal modes with frequencies 0.17 Hz and 0.19 Hz [Fig.

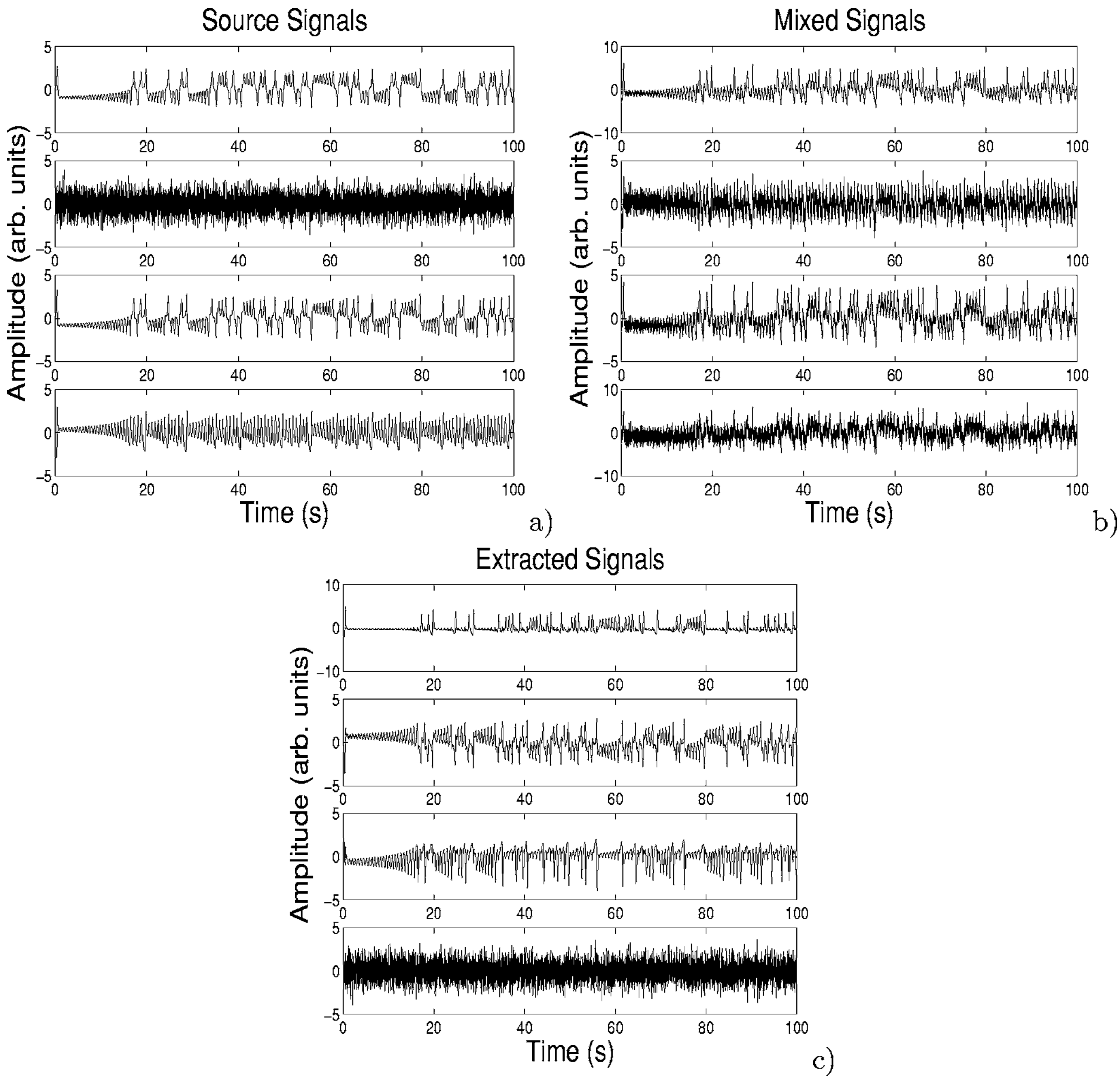

FIG. 8. Separation of signals from the mixtures of the three components of Lorenz system and Gaussian noise: (a) source signals; (b) mixed signals; (c) extracted signals.

2(c)]. The probability distribution and the autocorrelation indicate that the source noise as well as the extracted one is pure [Fig. 2(d)].

B. Piecewise linear dynamical systems

We have selected a particular piecewise linear oscillator: the Andronov oscillator [12]. This is the simplest system that generates, in a specific range of parameters, a dynamically stable limit cycle, which is approached asymptotically by all other phase paths. The equations of Andronov oscillator are

$$\ddot{x} + 2h_1\dot{x} + \omega_0^2 x = 0 \quad \text{if } x < b,$$

$$\ddot{x} - 2h_2\dot{x} + \omega_0^2 x = 0 \quad \text{if } x > b, \tag{8}$$

where b is the threshold which takes into account the nonlinearity of the system via a self-coupling. The analogical system representing the Andronov oscillator is the valve oscillator with the oscillating circuit in the anode circuit and inductive feedback. A simple realization is obtained by neglecting the anode conductance and assuming a piecewise linear approximation for the valve characteristic (sigmoid characteristic) $i_a = i_a(u)$, where u is the grid voltage and i_a is the anode current.

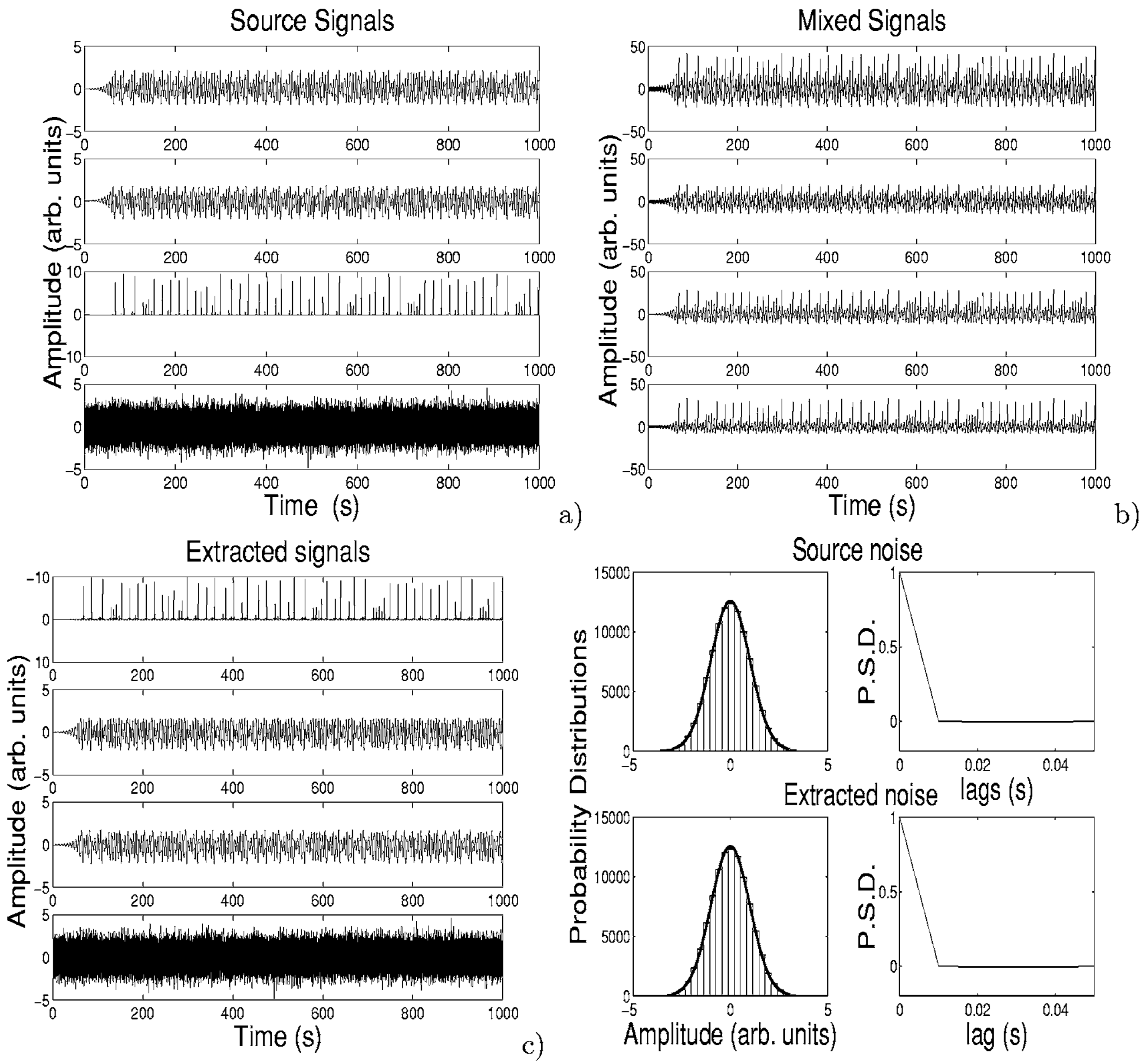

FIG. 9. Separation of signals from the mixture of the three components of Rossler system and Gaussian noise: (a) source signals; (b) mixed signals; (c) extracted signals; and (d) distribution function and autocorrelation of source and extracted noise.

The equations are given in dimensionless variables $x = u/u_0$, where $-u_0$ is the cutoff voltage, ω_0 is the natural frequency of the circuit, and h_1 and h_2 are, respectively, the dissipative and constructive parameters. The phase space is divided by the straight line $x=b$ into two different regions identified by the two differential equations of the system [Eq. (8)] [12].

By using different parameters and thresholds, the Andronov oscillator has different behaviors. In fact, if the threshold is negative (e.g., $b=-1$) we have a limit cycle only if $0<h_2<h_1<1$ otherwise forcing oscillations are installed, while if the threshold is positive the system is basically dissipative. In the following, we study also the system of two weak linearly coupled Andronov oscillators.

Experimental results: Andronov oscillator

We select the following examples to illustrate the interesting results by applying ICA to this system when generates self-sustained oscillations:

(1) the separation of the Andronov oscillators and additive Gaussian noise;

(2) the separation of two coupled Andronov oscillators, one Andronov oscillator, and a Gaussian noise; and

(3) the performance of ICA to separate these kinds of signals.

We begin the analysis considering as source signals three Andronov oscillators and Gaussian noise (Fig. 3). In Fig. 3(a), we show one of the three Andronov oscillators in the phase space. The parameters $h_1=1.3$, $h_2=1$, and $b=-1$ are

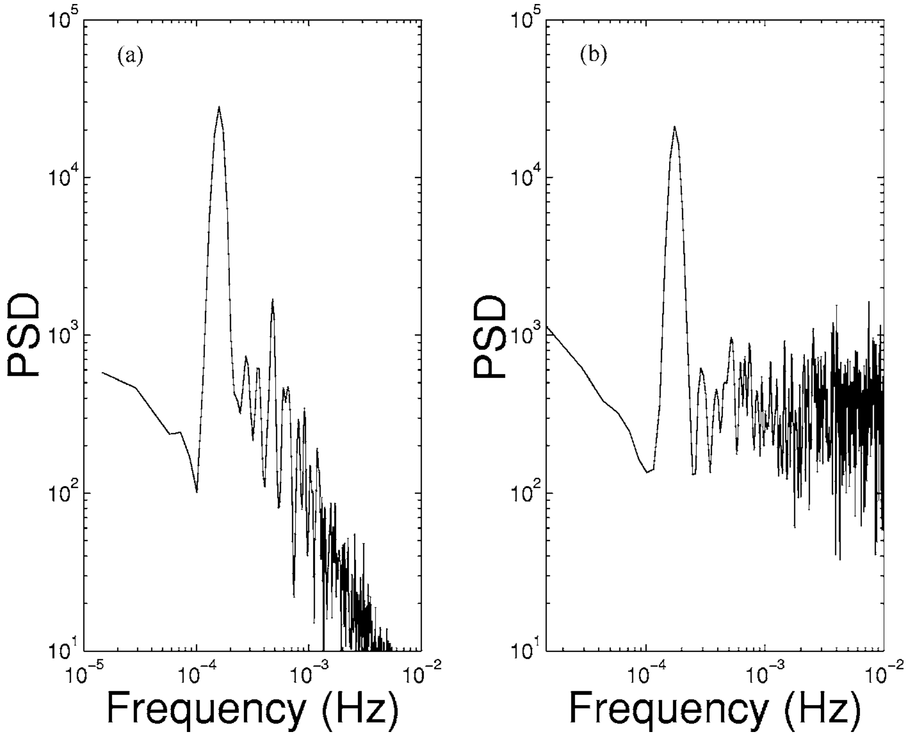

FIG. 10. Power spectrum density: (a) a generic story of stochastic process described by Eq. (12); and (b) $z(t)$ described by Eq. (13).

the same in all the experiments. The frequencies of the oscillators are 0.8 Hz, 0.9 Hz, and 1.1 Hz with a sampling frequency of 125 Hz. Applying ICA, we obtain a good separation as reported in Fig. 3(d), recovering the original self-sustained oscillations.

In the second experiment, we consider the first piecewise linear oscillator simply added to the others, which are coupled to give a nonlinear beating regime. The system is affected by Gaussian noise. In Figs. 4(a) and 4(b), the source signals and the mixtures are reported with the associated spectra. Applying the ICA, we extract four separated signals [Fig. 4(c)]: the three independent self-sustained oscillations (with the three different frequencies) and the added noise. Also in this case, the probability distribution and the autocorrelation, reported in Fig. 4(d), indicate that the source noise as well as the extracted one is pure.

In the third experiment, we show the performance of the ICA to separate a self-sustained oscillation (Andronov oscillator) by a Gaussian noise. In this case, we use an Andronov oscillator with frequency 0.8 Hz and we estimate the differences between the true frequency and the estimate one varying the SNR from −100 to 20 db. As in the linear case, ICA identifies the true frequency at a lower SNR than PSD (Fig. 5).

In conclusion, linearly coupled Andronov oscillators (piecewise linear oscillators) are well separated by ICA both among them and from superimposed noise. These experiments show the power of ICA that is able, as in the linear case with regards the normal modes, to extract the independent self-oscillations in time domain.

C. Nonlinear dynamical systems in limit cycle regime

We have selected a particular nonlinear dynamical system: the Van der Pol oscillator (see, for example [13]). This system, under certain conditions, undergoes a limit cycle regime. The equation that describes this system is

$$\ddot{x} + b(x^2 - 1)\dot{x} + \omega_0^2 x = 0, \tag{9}$$

where ω_0 is the frequency and b is a constant that affects how nonlinear the system is. For b equal to zero, the system is actually just a linear oscillator. As b grows the nonlinearity becomes impossible to ignore.

1. Experimental results: Van der Pol oscillator

The aim is to show that by ICA it is possible to extract the original wave forms in time domain, corresponding to the limit cycle regime, from weakly coupled Van der Pol systems.

In this experiment the parameter b is equal to 0.3 and the frequencies of the oscillators are 1.1 and 1.2 Hz with a sampling frequency of 125 Hz.

In Fig. 6(a), we report the source signals (nonlinear beating regime), while in Fig. 6(b), the independent signals are separated. We stress that it is not trivial because the nonlinear differential equations cannot be solved and fast fourier transform (FFT), due to the nonlinearity of the problem, looses its efficacy.

D. Chaotic dynamical systems

The studied chaotic systems are the Lorenz oscillator and the Rossler system [14]. The Lorenz oscillator was discovered by Lorenz in 1963 as a very simplified model of convection rolls in the upper atmosphere. Lorenz found that the trajectories of this system, for certain settings, do not diverge to infinity, and never settle down to a fixed point or to a stable limit cycle. The trajectories, instead, in the phase

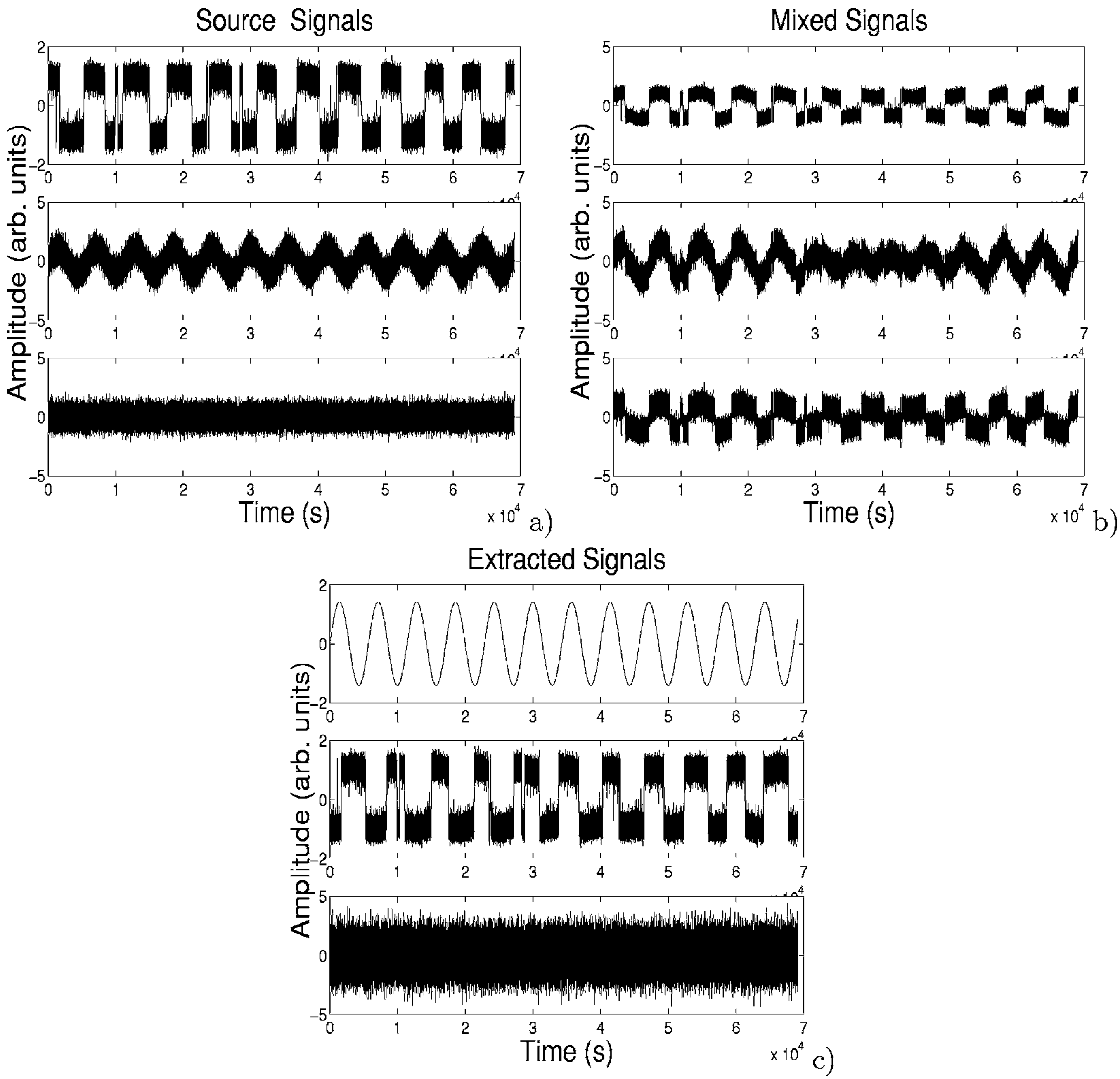

FIG. 11. Separation of the mixture of a signal generated by Eq. (13) (with Ω equal to 0.003 Hz), an additive Gaussian noise, and a signal described by Eq. (12) with the same resonance frequency: (a) source signals; (b) mixed signals; and (c) extracted components.

space, have a complex form: in this way Lorenz introduced the strange attractor and the concept of chaos. The Lorenz's system can be cast in the following form:

$$\dot{x} = -\sigma(x-y),$$

$$\dot{y} = -xz + rx - y,$$

$$\dot{z} = xy - bz, \qquad (10)$$

with σ, r, and b adimensional parameters.

Another example of deterministic chaos is the system proposed by Rossler. It arose from some works in chemical kinetics. The system is described by three coupled nonlinear differential equations:

$$\dot{x} = -y - z,$$

$$\dot{y} = x + ay,$$

$$\dot{z} = b + z(x-c), \qquad (11)$$

with a, b, and c adimensional parameters.

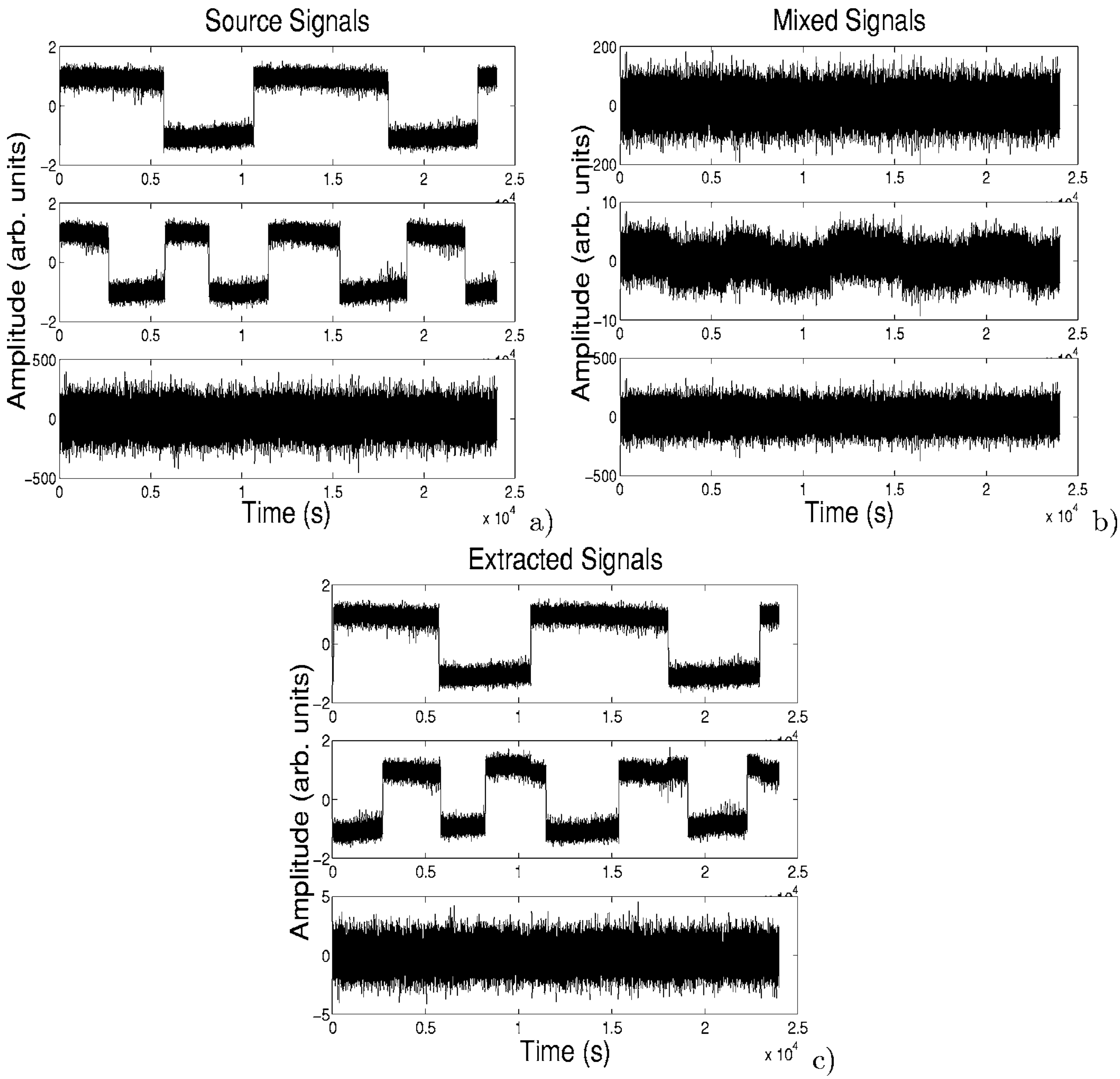

FIG. 12. Separation of the mixture of two signals in stochastic resonance regime and Gaussian noise: (a) source signals; (b) mixed signals; (c) extracted signals.

Experimental results: Lorenz oscillator and Rossler system

In this case we analyze, in detail, some representative examples to show:

(1) the separation of the three components of Lorenz oscillator and noise with uniform distribution;

(2) the separation of the three components of Lorenz oscillator and a Gaussian noise; and

(3) the separation of the three components of Rossler system and Gaussian noise.

We use this set of parameters for Lorenz system: $\sigma=10$, $r=28$, and $b=\frac{8}{3}$. In the first experiment (Fig. 7), ICA has extracted, in an optimal way, the components of Lorenz oscillator and the noise. If we also add one harmonic oscillator and one Andronov oscillator, the ICA separates all the involved DSs in a very good way.

The separation is not perfect when we consider mixtures of the components of Lorenz oscillator with Gaussian noise (Fig. 8). In this particular case, the recognition of the component z of the Lorenz oscillator is impossible because the component has a distribution very similar to a Gaussian distribution introducing the intrinsic ambiguities of ICA (two Gaussian variables are forbidden) [1].

Instead, in the case of the Rossler system ($a=0.15$, $b=0.2$, and $c=10$) the separation of three components and Gaussian noise is excellent as you can see in Fig. 9. The distribution function and autocorrelation of source and extracted noise in Fig. 9(d) show that nothing of spurious is present in the extracted noise, which is a white Gaussian noise.

E. Stochastic systems

The fourth class of systems that we describe is that of diffusive systems in the regime of stochastic resonance. Since its introduction [15], stochastic resonance has become very popular in many fields of natural science. The term is given to a phenomenon that is manifest in nonlinear systems. Although in the recent literature the notion of stochastic resonance gained broader significance, the archetype of system that can rule stochastic resonance regime is represented by a simple symmetric bistable potential (double-well) driven by both an additive random noise, i.e., white and Gaussian, and an external periodic bias. Given these features, the response of the system undergoes resonancelike behavior as a function of the noise level and of the parameters [15]; hence the name stochastic resonance [16]. Formally, we consider the Langevin equation with a small periodic forcing:

$$dx = [x(a - x^2) + A\cos(\Omega t)]dt + \varepsilon dW, \qquad (12)$$

where W is a Wiener process, i.e., a Gaussian process with zero mean and unitary variance; x is a not dimensional variable; A is the amplitude; and Ω the angular frequency of the external periodical forcing.

We select this system because its Fourier transform is characterized by a well defined peak at the resonance frequency, see Fig. 10(a).

In our experiments we analyze this kind of systems and we also make a comparison with another that can display a similar frequency content. The latter system, denoted as $z(t)$, is described by the following equations:

$$\ddot{s} = -\Omega^2 s,$$

$$dy = \nu dW,$$

$$z(t) = As(t) + By(t), \qquad (13)$$

where the first equation is a simple harmonic oscillator with angular frequency Ω; the second equation is a genuine Wiener process; and the third equation is the superposition of the two, according to the coefficient A, B. Choosing Ω equal to the resonance frequency of the system described by Eq. (12), we obtain that $z(t)$ has a similar frequency content as $x(t)$ [see Fig. 10(b)].

Experimental results: Diffusive systems in stochastic resonance regime

We made many experiments with this kind of system, here we want to show two very interesting results:

(1) the separation of a mixture of a periodic signal with Gaussian noise and a generic story from a diffusive process in stochastic resonance regime; and

(2) the separation of two generic stories from diffusive processes in stochastic resonance regime with different resonance frequencies and a Gaussian noise.

We begin our analysis just considering the performance of ICA in two cases with FFT seemingly similar but very different, namely a mixture of periodic signal and Gaussian noise, i.e., Eq. (13), and a generic evolution in the stochastic regime described by Eq. (12).

As we can see in Fig. 11, the separation is optimal; ICA recognizes low-dimensional and high-dimensional systems, i.e., harmonic oscillator and both the diffusive process in stochastic resonance regime and Gaussian noise, also in the presence of a similar frequency content. We should note that linear methods based on FFT fail because they do not distinguish the real number of degrees of the system underlying our mixtures, describing the observed spectra as due to the same DS. In that framework, a system with a stochastic resonance like behavior is not at all different from a simple oscillator with noise.

In the second experiment, we consider the mixtures of systems in stochastic resonance regime and Gaussian noise [Figs. 12(a) and 12(b)]. The relative resonance frequencies are 10^{-3} and 0.5×10^{-3} Hz with a sampling frequency of $\frac{1}{0.3}$ Hz. As a result we obtain similar signals as input [Fig. 12(c)].

The previous results have two important consequences: (1) ICA recognizes from the mixtures the two different resonancelike behaviors; and (2) it is not possible to extract the part of intrinsic noise present in Eq. (12) since it is dynamically superposed.

Summarizing these experiments are very impressive because they assure that, if we have real signals, which are linear superposition of linear, nonlinear, and stochastic processes, ICA is able to identify them, giving to us their wave forms in time domain and a very clear indication about the complexity of the dynamical systems involved in our data. This is extremely important when we want to construct any physical model to explain the observed phenomenon.

IV. CONCLUSIONS

Now we can draw our conclusions. We have applied ICA to very representative dynamical systems; first to linear and nonlinear systems with few degrees of freedom, and then to infinite degrees of freedom systems, namely, diffusion processes in the regime of stochastic resonance.

Regarding the linear systems, we obtained very good separation from very high superimposed noise, with SNR ranging from −100 db to 20 db. Furthermore, the ICA acts as a fast Fourier transform but in time domain, in separating coupled oscillators, since it gives us the normal modes of the system. It is remarkable that it is not simple to extract periodic system from noise and that if we consider the case of coupled oscillators when the frequencies are commensurable, standard methods as, for example, Grassberger and Procaccia analysis are not able to individuate the right phase-space dimensions (degrees of freedom). ICA solves the first problem extracting a periodic signal from a noise with an amplitude 1000 times higher and the second one giving the normal modes that represent the real degrees of freedom of the system. The performance of ICA is valuable also in the case of piecewise linear systems: in this case we separate coupled

Andronov oscillators that are highly nontrivial self-coupled dynamical systems. Also in these experiments the separation from noise is well made.

Going to full nonlinear systems, we consider the case of the Van der Pol oscillator generating limit cycles and we obtain the extractions of limit cycle wave forms in time domain. In the case, then, of chaotic DSs, the ICA gives good separations affirming, clearly, to be a technique able to identify any type of DS independently from its complexity. We note that the perfomance of ICA in these two last cases loses its efficacy at lower SNR than in the linear cases.

The experiments with stochastic resonance are very impressive: the superimposed periodic and stochastic signals are completely separated, i.e., ICA perfectly recognizes the different superimposed dynamical systems also when the Fourier transform is irresolute (insensitive). As a conclusive remark we can say that ICA is a very good method of preanalysis for scalar series, namely it allows one to recognize if the scalar series contain one or more independent dynamical systems. The results contained in this paper are the abstract generalization of our efforts in applying ICA to a lot of natural physical systems, i.e., organ pipes [17], atmospheric environment [4], and volcanic systems [2,3,5].

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